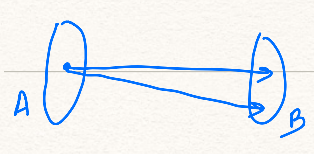
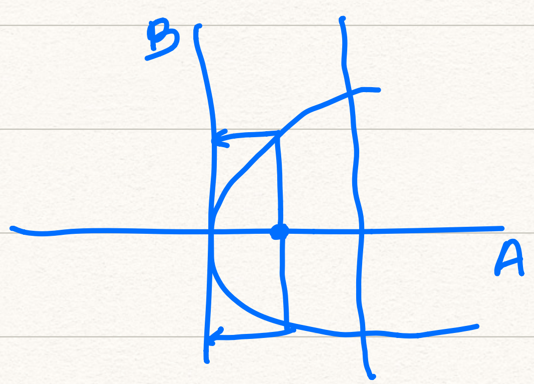
Notation and Definitions

- A <u>function</u> from a set A to a set B (written as $f: A \to B$) defines a rule which assigns to each $x \in A$ a unique element $y \in B$.

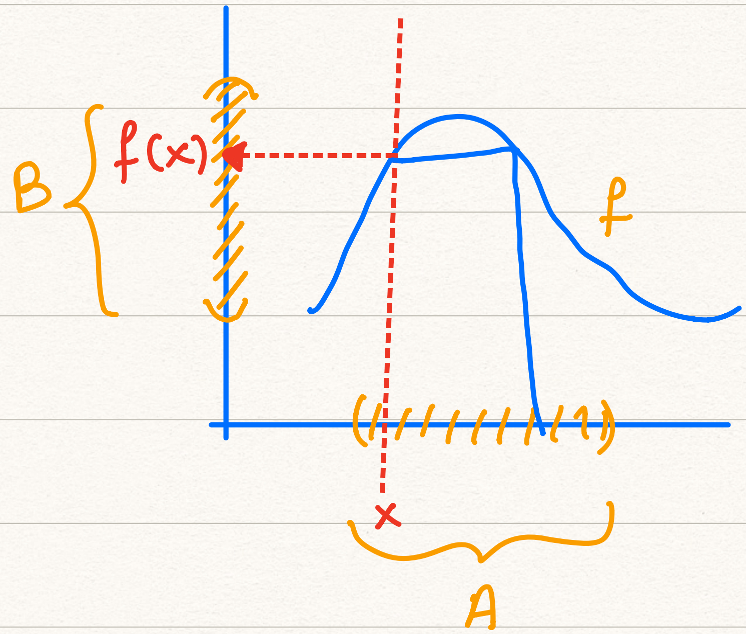
  The element y is called the image of the element x and we write $y = f(x)$.

  If either the rule f or the set A or the set B are changed, then we will consider it a different function.

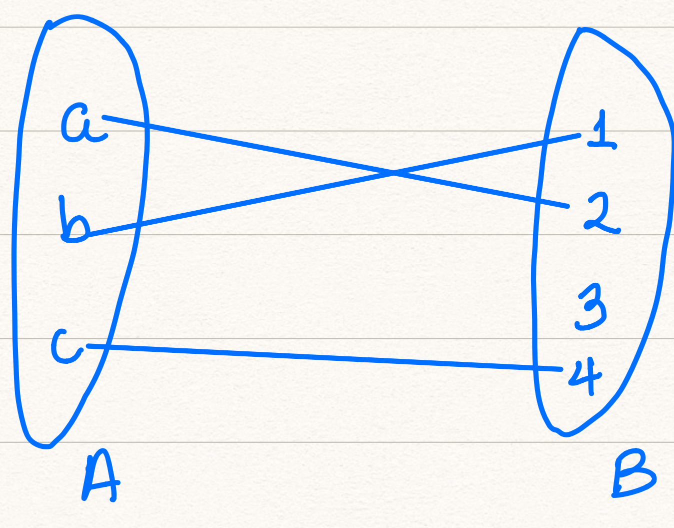




- When A and B are sets of real numbers, we can draw the graph of the function

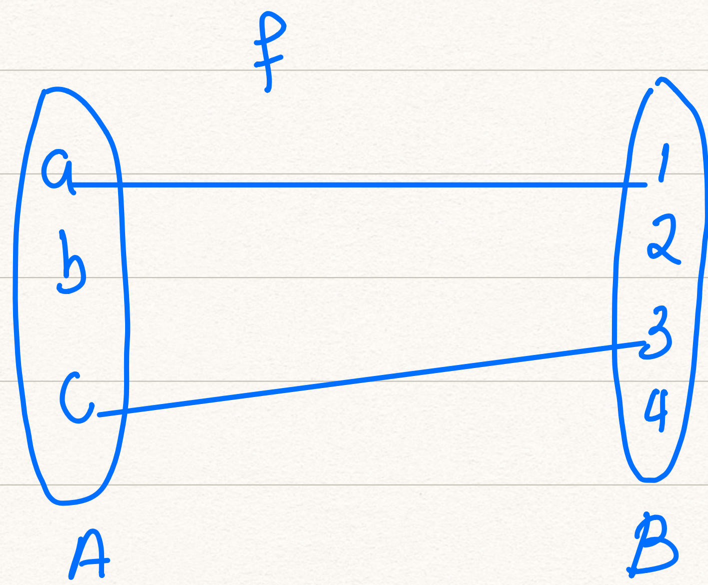


Example:



This is a function

Example.



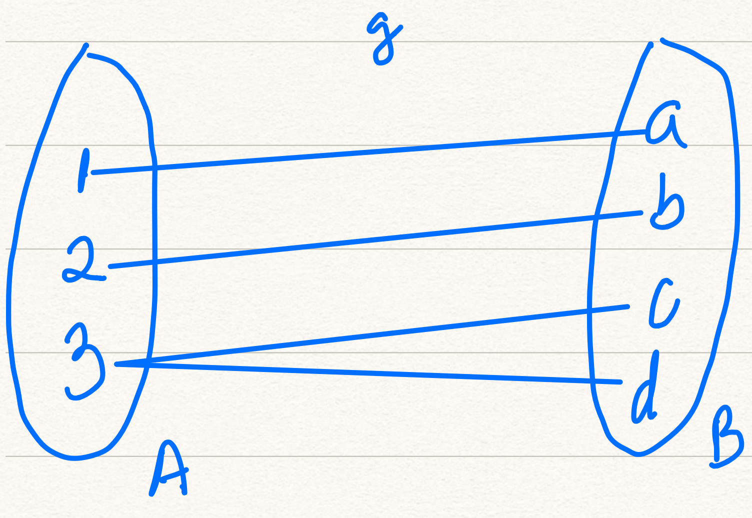
Not a function

$f : A \to B$

$\tilde{A} = \{a, c\}$

is $f : \tilde{A} \to B$ a function? yes

Example.



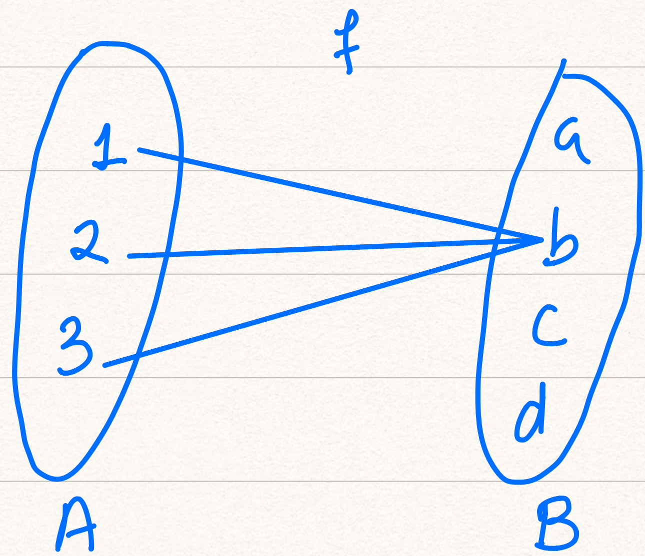
$g : A \to B$ is not a function.

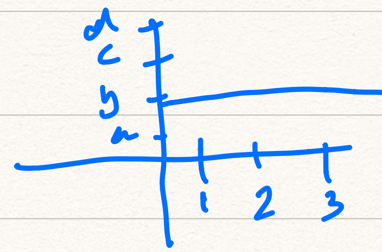
$\tilde{B} = \{a, b, c\}$

$g : A \to \tilde{B}$

is a function

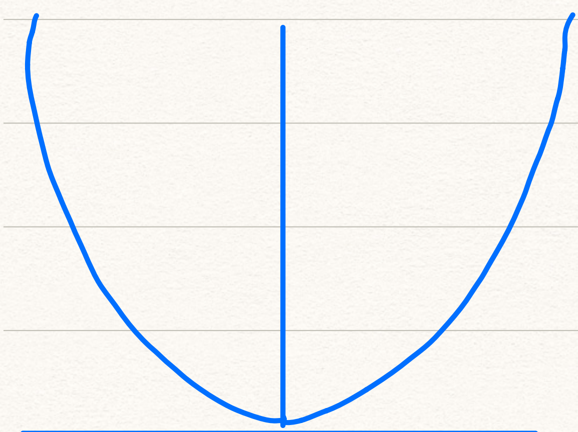
Example





$f(x) = b$

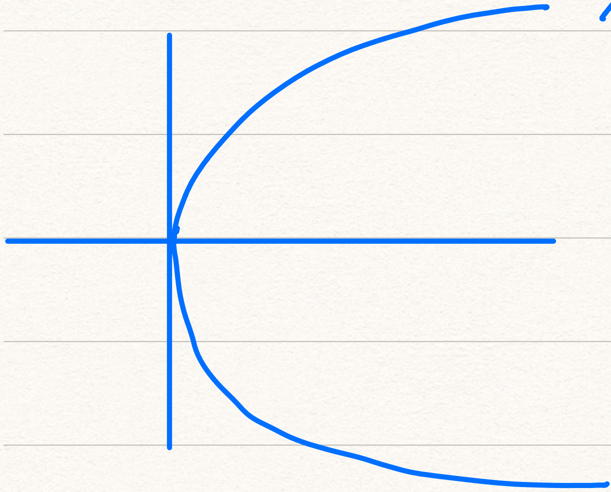
Example



$f: \mathbb{R} \to \mathbb{R}$

$f: \mathbb{R} \to [0, \infty)$

Example



✗ $g_0 : \mathbb{R} \to \mathbb{R}$

✗ $g_1 : [0, \infty) \to \mathbb{R}$

✓ $g_2 : [0, \infty) \to [0, \infty)$

- Let $f : A \to B$ and $S \subseteq A$. We define the set

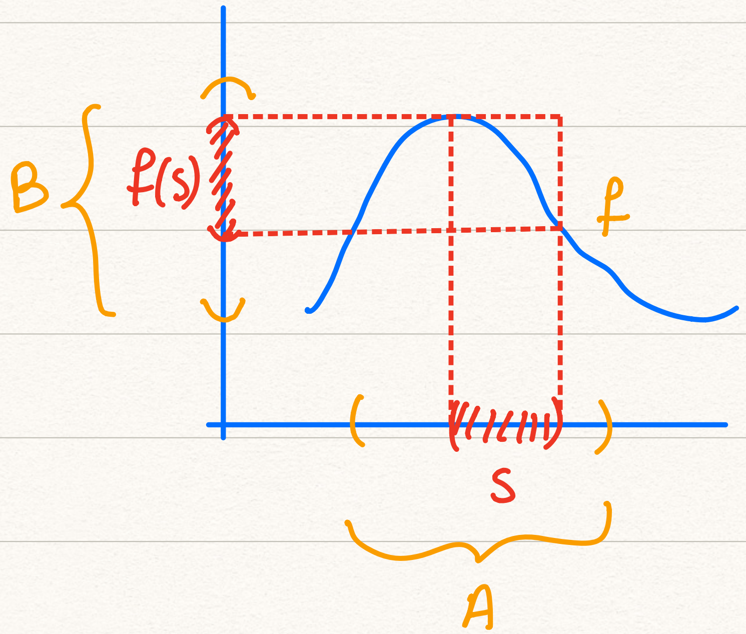
$$f(S) = \{ f(x) : x \in S \}$$

$f(S)$ is called the image of S under f.

a set

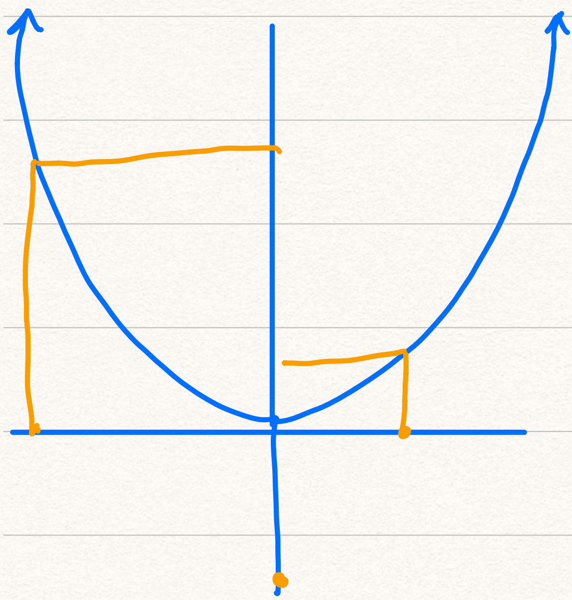
$$y = f(x)$$

an element



- Let $f: A \to B$. The set A is called the domain of f and $f(A) \subseteq B$ is called the range of f.

Example



$f: \mathbb{R} \to \mathbb{R}$

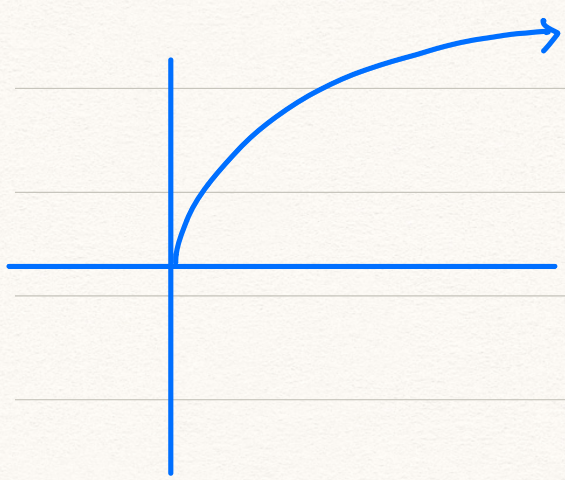
Domain: $\mathbb{R}$

Range: $[0, \infty)$

$y < 0$

$y = f(x) = x^2$

Example



$g: [0, \infty) \to \mathbb{R}$

Domain: $[0, \infty)$

Range: $[0, \infty)$

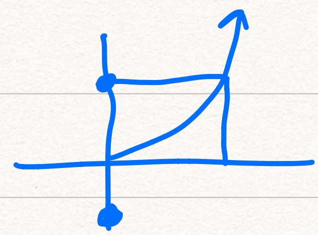
$f(S)$, $f(x)$

$\uparrow$

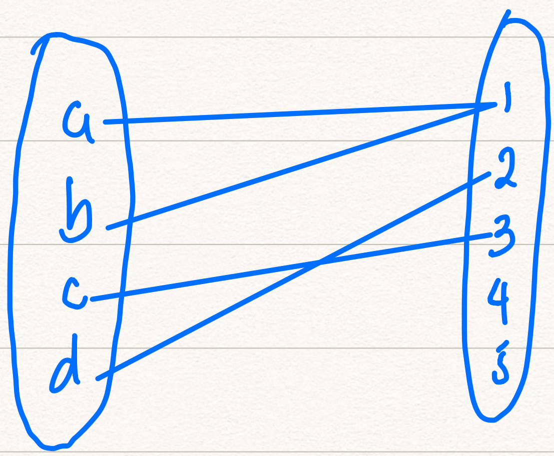
A

Definition

A function is injective, or one-to-one, if for every pair of numbers $x_1 \neq x_2$ we have $f(x_1) \neq f(x_2)$. If a function is injective, the equation $y = f(x)$ has either no solution or a unique solution.



Example



f

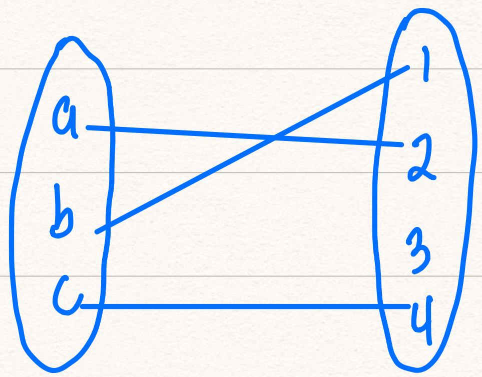
$1 = f(x)$

$x = a, \quad x = b$

$a \neq b$ but $f(a) = f(b)$

Not injective.

Example



f

$y = f(x)$

$3 = f(x)$ No solution

$1 = f(x)$ $x = b$

$2 = f(x)$ $x = a$

$4 = f(x)$ $x = c$

It is injective

Example

$y = x^2$

$x^2 = 1$

$x = -1, \; x = 1$

Not injective

Example

$f(x) = 5x + 3$ bijective.

For any specific value of $y \in \mathbb{R}$,

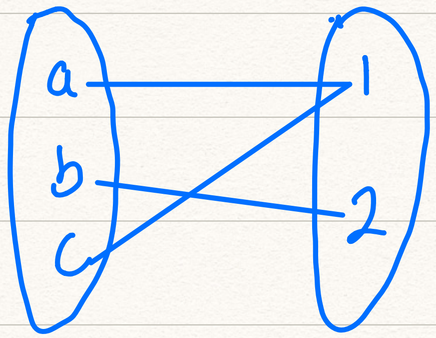
$$x = \frac{y-3}{5}$$

Injective.

Definition

A function $f: A \to B$ is surjective, or onto, if $f(A) = B$. If a function is surjective, the equation $y = f(x)$ always has at least one solution for each $y \in B$.

Example.



Injective: No

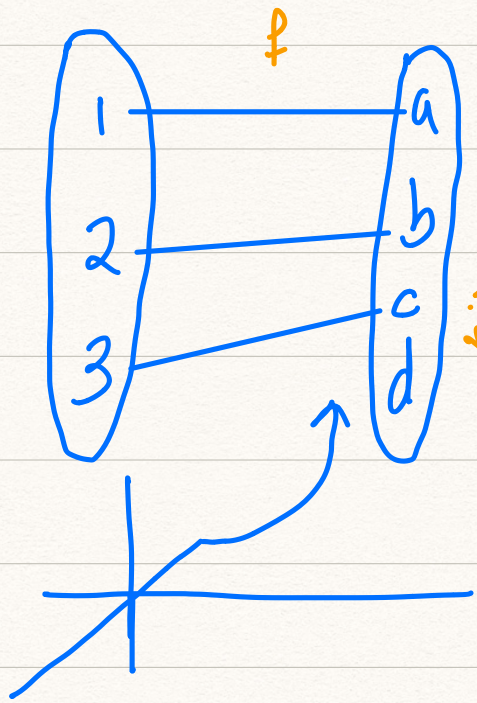
Surjective- yes

$1 = f(x)$ $x = a$, $x = c$

$2 = f(x)$ $x = b$

Example

$y = f(x)$



Injective: yes

Surjective: No

bijective $\rightarrow f: A \rightarrow \tilde{B}$ where $\tilde{B} = \{a, b, c\}$

$a = f(x)$ $x = 1$

$b = f(x)$ $x = 2$

$c = f(x)$ $x = 3$

$\rightarrow d = f(x)$ No solution

Example

$f(x) = x^2$

$f: \mathbb{R} \to \mathbb{R}$

$f: \mathbb{R} \to [0, +\infty)$

$f: [0, +\infty) \to [0, +\infty)$
bijective.

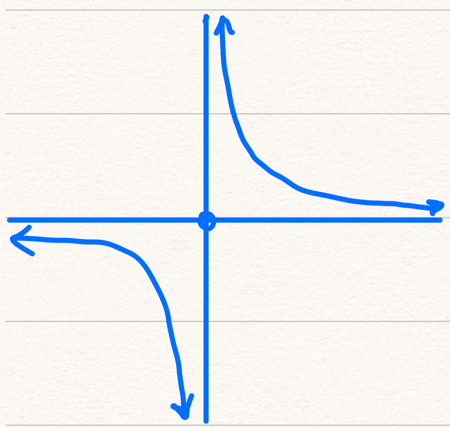
$f: \mathbb{R} \to \mathbb{R}$ Injective No

$-5 = x^2$ surjective No

$f: \mathbb{R} \to [0, +\infty)$ surjective Yes $y = f(x)$

Example:

$0 = \frac{1}{x}$



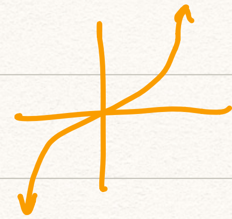
$f: \mathbb{R} \to \mathbb{R}$ Surjective: No, Injective: Yes } Not a function

$f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$ surjective. No

- $f: \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ surjective

bijective.

$y = x^3$



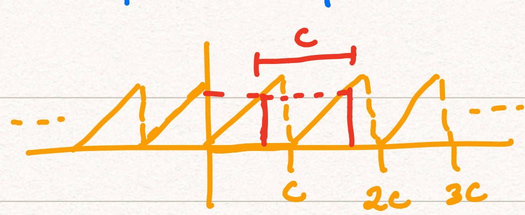
Definition

A function is bijective if it is both injective and surjective.

If a function is bijective, the equation $y=f(x)$ always has a unique solution for each $y \in B$.

Definition

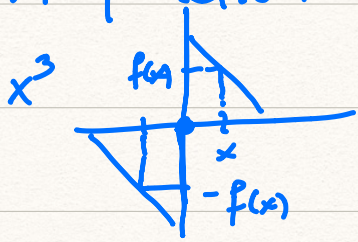
A function is periodic if there exists some $c>0$ such that $f(x+c)=f(x)$. The smallest such c is referred to as the period of the function.

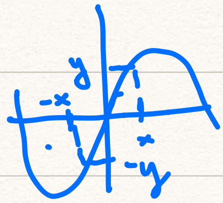


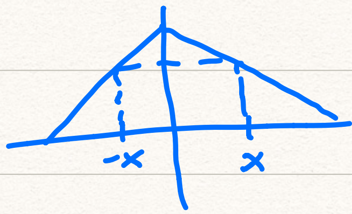
Definition

A function is even if $f(-x)=f(x)$.

A function is odd if $f(-x)=-f(x)$.

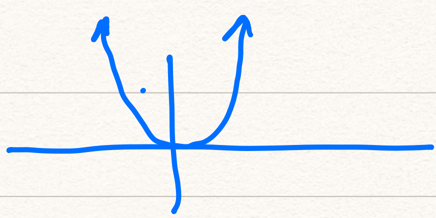


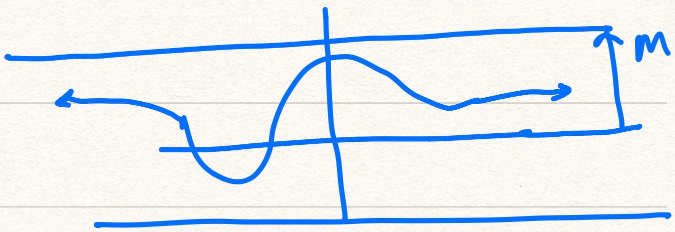




Definition

A function is bounded if there exists some $M>0$ such that $|f(x)|\leq M$ for all x in its domain.

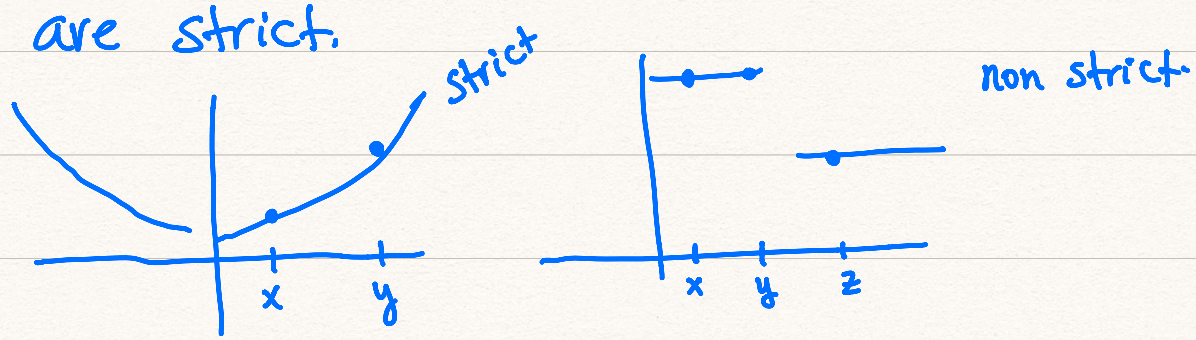




Definition

A function is montonically increasing if for every x,y in its domain such that $x<y$ it satisfies $f(x)\leq f(y)$, and monotonically decreasing if $f(x)\geq f(y)$.

We say that it is monotonic strictly increasing/decreasing if the inequalities are strict.



Elementary Functions

$\ln e^x = x$

See summary in AV.

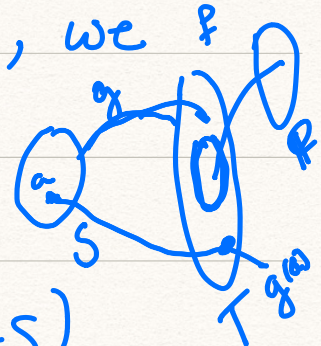
Combining functions

- For $f, g : A \to \mathbb{R}$,
  - $(f+g)(x) = f(x) + g(x) \quad (x \in A)$
  - $(\lambda f)(x) = \lambda f(x) \quad (x \in A, \lambda \in \mathbb{R})$

- $(fg)(x) = f(x)g(x) \quad (x \in A)$

- $(f/g)(x) = \dfrac{f(x)}{g(x)} \quad (x \in A,\ g(x) \neq 0).$

▶ For $g: S \to T$ and $f: T \to \mathbb{R}$, we define $f \circ g : S \to \mathbb{R}$ by



$$f \circ g(x) = f(g(x)) \quad (x \in S)$$

$e^{\ln x}$ ✓ $\qquad f = e^x : \mathbb{R} \to (0, +\infty) \qquad f(g(a))$

$\ln e^x$ ✓ $\qquad g = \ln x : (0, +\infty) \to \mathbb{R}$

<u>Example</u>. Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

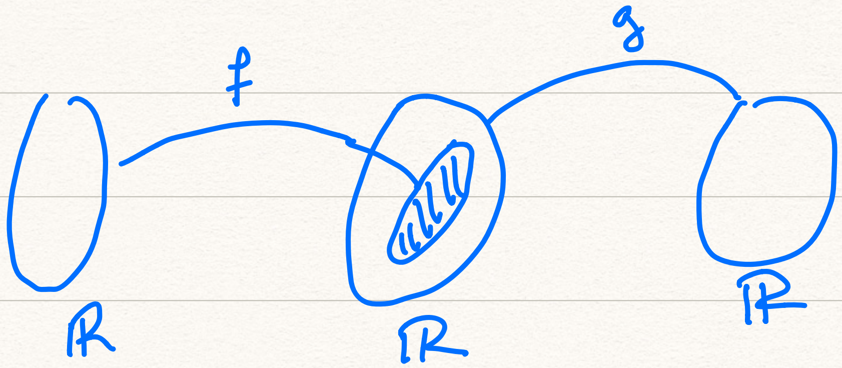
$$f(x) = \frac{x^2 - 1}{x^2 + 1} \qquad x^2 + 1 = 0$$

and let $g: \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = x^3$$

$f \circ g$ and $g \circ f$ $\qquad f \circ g \neq g \circ f$

$$f \circ g(x) = f(g(x)) = \frac{(g(x))^2 - 1}{(g(x))^2 + 1} = \frac{x^6 - 1}{x^6 + 1}$$



$$g \circ f(x) = g(f(x)) = (f(x))^3 = \left(\frac{x^2 - 1}{x^2 + 1}\right)^3$$

Inverse functions

- We say that f^{-1} is the inverse function to $f: A \to B$ if f^{-1} is a function from B to A which has the property that $x = f^{-1}(y)$ if and only if $y = f(x)$.

$$Id(x) = x = f^{-1}(f(x)) \qquad Id(x) = x$$

- Not all functions have an inverse. In fact, f has an inverse if and only f is bijective.

Example. Consider the function

$$f(x) = \frac{x-1}{x+1}$$

Domain: $\mathbb{R} \setminus \{-1\}$

Range: $\mathbb{R} \setminus \{1\}$

$$f(x) = y = \frac{x-1}{x+1} \rightarrow y(x+1) = x-1$$

$$\rightarrow xy + y = x - 1$$

$f(-1)$ ✗

$$xy - x = -y - 1$$

$$x(y-1) = -y-1$$

$$x = \frac{-y-1}{y-1} = f^{-1}(f(x))$$

$$f^{-1}(x) = \frac{-x-1}{x-1}$$ ✓

Domain f^{-1} = Range f

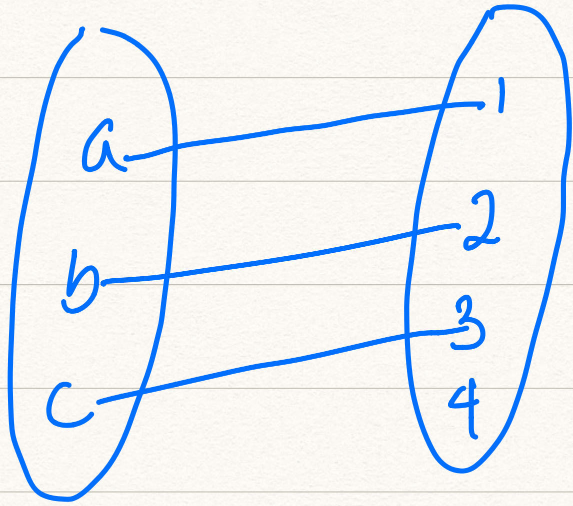
$\mathbb{R} \setminus \{1\}$

$$f^{-1}(f(x)) = \frac{-f(x)-1}{f(x)-1} = \frac{-\left(\frac{x-1}{x+1}\right)-1}{\frac{x-1}{x+1}-1}$$

$$= \frac{\frac{-(x-1)-(x+1)}{\cancel{x+1}}}{\frac{x-1-(x+1)}{\cancel{x+1}}}$$

$$= \frac{-x+1-x-1}{x-1-x-1} = \frac{-2x}{-2} = x$$

$$1 = \frac{x-1}{x+1} \Rightarrow \cancel{x}+1 = \cancel{x}-1 \Rightarrow 1 = -1 \quad X$$



$1 = f(a)$

if and only if

$a = f^{-1}(1)$

$4 = f(x)$

if and only if

$x = f^{-1}(4)$

$e^x : \mathbb{R} \to (0, +\infty)$ $\quad 0 = e^x$

$\to y = e^x$

$\ln x : (0, +\infty) \to \mathbb{R}$

Post Chapter 2 problems in AV.